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ON SMOOTH, NONLINEAR SURJECTIONS OF BANACH SPACES

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ABSTRACT

It is shown that (1) every infinite-dimensional Banach space admits a C^1 Lipschitz map onto any separable Banach space, and (2) if the dual of a separable Banach space X contains a normalized, weakly null Banach-Saks sequence, then X admits a C^∞ map onto any separable Banach space. Subsequently, we generalize these results to mappings onto larger target spaces.

Introduction

By the classical Morse-Sard theorem (see [13]), a smooth surjective mapping of one euclidean space onto another must be submersive at some point, i.e., its derivative at some point must itself be surjective. If X, Y are infinite-dimensional Banach spaces, continuous *linear* mappings of X onto Y do not exist in general; thus it is natural to ask whether X nevertheless admits a nonlinear transformation onto Y.

A special case of a theorem from [3] implies that an infinite-dimensional Banach space X admits a Lipschitz surjection onto any Banach space Y for which $\mathrm{dens}(X) \geq \mathrm{dens}(Y)$. (The *density character* $\mathrm{dens}(V)$ of a metric space

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V is the smallest cardinality that a dense subset of V can have.) The purpose of the present article is to establish the existence of smoother surjections under somewhat more restrictive assumptions.

THEOREM 1: *Every infinite-dimensional Banach space X admits a C^1 Lipschitz map onto any separable Banach space.*

Recall that a weakly null sequence (x_j) in a Banach space X is called *Banach-Saks* if, for any subsequence (y_j) of (x_j), the sequence of arithmetic means

$$\frac{1}{n}\Big\|\sum_{j=1}^{n} y_j\Big\|$$

tends to 0 as $n \to \infty$. An example of a weakly null Banach-Saks sequence is provided by the standard basis of any $\ell_p(\mathbb{N})$ space for $1 < p < \infty$.

THEOREM 2: *If X^* contains a normalized, weakly-null Banach-Saks sequence, then X admits a C^∞ mapping onto any separable Banach space.*

For mappings onto larger target spaces, we introduce the following terminology. Given a cardinal $\mathcal{K}$, we say that a Banach space X satisfies condition $(\mathcal{K})_1$ provided that there exists a Banach space X' which admits a C^1 Lipschitz bump function, a collection $(x_\alpha)_{\alpha\in\Gamma} \subset X$ with $\operatorname{card}(\Gamma) = \mathcal{K}$, and a bounded linear map $T: X \to X'$ such that the vectors $(Tx_\alpha)_{\alpha\in\Gamma}$ are ε-separated for some $\varepsilon > 0$.

We will say that a Banach space X satisfies condition $(\mathcal{K})_\infty$ provided that there is a normalized collection $(x^*_\alpha)_{\alpha\in\Gamma} \subset X^*$ such that $\operatorname{card}(\Gamma) = \mathcal{K}$ and for each $\varepsilon > 0$, there exists $k(\varepsilon)$ such that $\operatorname{card}\{\alpha \in \Gamma: |x^*_\alpha(x)| > \varepsilon\} \leq k(\varepsilon)$ for each $x \in X$ with $\|x\| \leq 1$.

In particular, the space $c_0(\Gamma)$ satisfies condition $(\mathcal{K})_1$ whenever $\operatorname{card}(\Gamma) \geq \mathcal{K}$, and any superreflexive space X with $\operatorname{dens}(X) \geq \mathcal{K}$ satisfies condition $(\mathcal{K})_\infty$.

THEOREM 3: *Let X be a Banach space, let $\mathcal{K}$ be a cardinal number, and let Y be a Banach space with* $\operatorname{dens}(Y) \leq \mathcal{K}$.

1. *If X satisfies condition $(\mathcal{K})_1$, then X admits a C^1 Lipschitz mapping onto Y.*
2. *If X satisfies condition $(\mathcal{K})_\infty$, then X admits a C^∞ mapping onto Y.*

To place these results in perspective, we recall that Kadec [10] and Torunczyk [14] have proven that if X, Y are infinite-dimensional Banach spaces, then $\operatorname{dens}(X) = \operatorname{dens}(Y)$ precisely when X is homeomorphic to Y, i.e., when there exists a continuous bijection $X \to Y$ with continuous inverse. In general, however, such mappings cannot be very smooth; for example, the absence of C^k bump functions on a space X implies that there is no proper C^k map $X \to \ell_2(\mathbb{N})$. Finally, we recall that it is presently unknown whether every infinite-dimensional Banach space admits a continuous linear map onto a separable infinite-dimensional Banach space (see [11, p. 12]).

The surjections $f: X \to Y$ constructed in the proofs of our theorems satisfy $\operatorname{rank}(Df) \leq 1$ at all points of X (compare [1, 4]). This rank restriction, although interesting, is evidently not a necessary condition for smooth surjections *per se*; nevertheless it reflects the fact that the Fréchet derivative of any smooth mapping between certain Banach spaces must be highly singular at each point. For example, it is well-known that for $1 < p < r < \infty$, every bounded linear operator $\ell_r \to \ell_p$ is compact and thus approximable in the norm topology by finite rank operators. Consequently, no smooth surjection $\ell_r \to \ell_p$ is significantly more "efficent" (in the sense of [2], for example) than the rank-1 map given by Theorem 3.

The proof of Theorem 3(1) extends our previous use of bump functions to construct smooth surjections in [4]. To prove Theorem 3(2), we develop in Section 1 a suitable replacement for such functions using the geometric requirement of condition $(\mathcal{K})_\infty$. The main proof follows in Section 2, and we derive Theorems 1 and 2 from Theorem 3 in Section 3. We conclude with some examples in Section 4.

Throughout this paper, all Banach spaces under consideration are assumed to be real and infinite-dimensional. For a Banach space X we denote by $B_X(r)$ the ball of radius r centered at the origin of X. For a set Γ, we denote by $c_0(\Gamma)$ the Banach space of all functions $\gamma: \Gamma \to \mathbb{R}$ such that for each $\varepsilon > 0$, the set $\{\alpha \in \Gamma: |\gamma(\alpha)| > \varepsilon\}$ is finite. For $1 \leq p < \infty$, we denote by $\ell_p(\Gamma)$ we denote the space of p-summable elements of $c_0(\Gamma)$ with norm $\|\gamma\| = (\sum_{\alpha\in\Gamma} |\gamma(\alpha)|^p)^{1/p}$. Finally, we use the notation $\mathbb{Z}_+$ and $\mathbb{N}$ to distinguish between the positive and non-negative integers, respectively.

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1. Condition $(\mathcal{K})_\infty$ for Banach spaces

Consider a Banach space X with dual X^* endowed with the standard dual norm. A subset $(x_\alpha^*)_{\alpha\in\Gamma} \subset X^*$ is said to be **semi-normalized** if $0 < \inf_\alpha \|x_\alpha^*\| \leq \sup_\alpha \|x_\alpha^*\| < \infty$ and **normalized** if $\|x_\alpha^*\| = 1$ for all $\alpha \in \Gamma$. Throughout this section, we will assume that X satisfies condition $(\mathcal{K})_\infty$ and let $(x_\alpha^*)_{\alpha\in\Gamma} \subset X^*$ be a normalized collection with the property that for each $\varepsilon > 0$, there is a number $k(\varepsilon)$ such that $\operatorname{card}\{\alpha \in \Gamma\colon |x_\alpha^*(x)| > \varepsilon\} \leq k(\varepsilon)$ for all $x \in B_X(1)$. The starting point for our construction will be the following lemma.

LEMMA 1.1: *For each $\varepsilon > 0$ there exist a subset $\Gamma' \subset \Gamma$ and a semi-normalized collection $(x_\alpha)_{\alpha\in\Gamma'} \subset X$ such that*

1. $\operatorname{card}(\Gamma') = \mathcal{K}$,
2. $x_\alpha^*(x_\alpha) = 1$ *and* $|x_\alpha^*(x_\beta)| < \varepsilon$ *for all* $\alpha, \beta \in \Gamma'$ *with* $\alpha \neq \beta$.

Proof: Let $(x_\alpha)_{\alpha\in\Gamma}$ be any semi-normalized collection satisfying $x_\alpha^*(x_\alpha) = 1$ for all $\alpha \in \Gamma$. For each $\alpha \in \Gamma$, we define

$$S_\alpha = \{\beta \in \Gamma\colon \beta \neq \alpha \text{ and } |x_\beta^*(x_\alpha)| \geq \varepsilon\}.$$

By assumption, there exists an integer $K > 0$ such that $\operatorname{card}(S_\alpha) \leq K$ for all $\alpha \in \Gamma$. For any subset $J \subset \Gamma$, we set $S_J = \bigcup_{\alpha\in J} S_\alpha$. We will call a subset $I \subset \Gamma$ **good** provided that $I \cap S_I = \emptyset$. Evidently, each singleton $\{\alpha\} \subset \Gamma$ is good, and the union of an increasing nested sequence of good sets is good. By Zorn's lemma, there exist maximal good subsets of Γ.

Let I_0 be any maximal good subset of Γ, and, proceeding inductively, let I_k be a maximal good subset of

$$\Gamma \setminus \bigcup_{i=0}^{k-1} I_i$$

for $k = 1, \ldots, K$. A good subset $I \subset \Gamma$ is maximal precisely when $I \cap S_\alpha \neq \emptyset$ for all $\alpha \notin I \cup S_I$. Since $\operatorname{card}(S_J) \leq \operatorname{card}(J)$ for any infinite $J \subset \Gamma$, it follows that either $\Gamma \setminus \bigcup_{i=0}^{K}(I_i \cup S_{I_i})$ or at least one of the I_i is a good subset of Γ with cardinality $\mathcal{K}$, which we can take as Γ'. ∎

The basic cube $Q \subset X$ corresponding to the collection $(x_\alpha^*)_{\alpha\in\Gamma}$ is defined as the set

$$Q = \{x \in X\colon |x_\alpha^*(x)| \leq 1 \text{ for all } \alpha \in \Gamma\}.$$

It is easy to check that our condition on the x_α^* requires that Q be an unbounded subset of X.

Next, we associate a smooth function $h\colon X \to \mathbb{R}$ to $(x^*_\alpha)_{\alpha\in\Gamma}$ as follows. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a fixed C^∞ function such that $\varphi = 1$ on $[-1,1]$ and $\varphi(x) = 0$ when $|x| \geq 2$. Define

$$h(x) = \prod_{\alpha\in\Gamma} \varphi(x^*_\alpha(x)).$$

By the property satisfied by the x^*_α, only finitely many terms in the above product differ from 1 locally, and so h is well-defined and smooth. Note furthermore that $h = 1$ on the basic cube Q, while $h(x) = 0$ if $|x^*_\alpha(x)| \geq 2$ for any $\alpha \in \Gamma$. Additionally, the derivatives of the function h are bounded on bounded subsets of X. More precisely:

LEMMA 1.2: *The function h defined above has the property that for each $n, m \in \mathbb{N}$ the C^m norm of $h|_{B_X(n)}$ is bounded.*

Proof: Since the collection $(x^*_\alpha)_{\alpha\in\Gamma}$ is normalized, any linear map from X into euclidean space $\mathbb{R}^k$ of the form

$$x \mapsto (x^*_{\alpha_1}(x), x^*_{\alpha_2}(x), \ldots, x^*_{\alpha_k}(x))$$

has norm $\leq \sqrt{k}$ regardless of the choice of $\alpha_i \in \Gamma$. Now fix $n \in \mathbb{Z}$ and note that for any $x \in B_X(n)$, our choice of the x^*_α implies that

$$\operatorname{card}\{\alpha \in \Gamma\colon |x^*(x)| \geq 1\} = \operatorname{card}\{\alpha \in \Gamma\colon |x^*(x/n)| \geq 1/n\} \leq k(1/n).$$

From the equicontinuity of the collection $(x^*_\alpha)_{\alpha\in\Gamma}$, it follows that each $x \in B_X(n)$ has a neighborhood U such that for all but at most $k(1/n)$ indices α, we have $\sup_U |x^*_\alpha| \leq 1$.

Now define $g_k\colon \mathbb{R}^k \to \mathbb{R}$ by $g_k(y_1, \ldots, y_k) = \prod_{i=1}^k \varphi(y_i)$. From the preceding remarks, it follows that on the ball $B_X(n) \subset X$, the function h can be represented locally as the composition of a linear map $X \to \mathbb{R}^{k(1/n)}$ of norm $\leq \sqrt{k(1/n)}$ with the function $g_{k(1/n)}$. Since $g_{k(1/n)}$ has bounded support, its $\mathbf{C}^m$-norm is bounded and our assertion follows.

2. Proof of Theorem 3

In this section we give a detailed proof of Theorem 3(2) and sketch the similar and easier proof of Theorem 3(1).

Proof of Theorem 3(2): Let X be a Banach space and $(x_\alpha^*)_{\alpha\in\Gamma} \subset X^*$ a normalized collection such that for each $\varepsilon > 0$, there exists a number $k(\varepsilon)$ satisfying

$$\operatorname{card}\{\alpha \in \Gamma\colon |x_\alpha^*(x)| > \varepsilon\} \leq k(\varepsilon)$$

for all $x \in B_X(1)$. By Lemma 1.1, we may furthermore assume that there exists a collection $(x_\alpha)_{\alpha\in\Gamma} \subset X$ such that

$$x_\alpha^*(x_\alpha) = 1 \quad \text{and} \quad |x_\beta^*(x_\alpha)| < \tfrac{1}{5}$$

for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Finally, we note that X satisfies the conclusion of Theorem 3(2) whenever one of its quotient spaces does. Thus, we may assume that the collection $(x_\alpha^*)_{\alpha\in\Gamma}$ is separating, i.e.,

$$\bigcap_{\alpha\in\Gamma} \ker(x_\alpha^*) = 0.$$

Let Y be a Banach space with $\operatorname{dens}(Y) \leq \operatorname{card}(\Gamma)$. To prove the theorem, we will explicitly construct a C^∞ surjection $X \to Y$. Set $M = \sup_\alpha \|x_\alpha\| \geq 1$, fix a positive constant $\varepsilon < (8M)^{-1}$, and let $T_\alpha\colon X \to X$ be the affine dilation $T_\alpha(x) = \varepsilon\,(x + 5\,x_\alpha)$ for each $\alpha \in \Gamma$. By our choice of ε, the cubes $Q_\alpha = T_\alpha(Q)$, where Q is the basic cube defined in Section 1, are then pairwise disjoint and lie in the interior of Q. Let $\mathcal{Q}_0 = \{Q\}$, and for $k \geq 1$, define collections $\mathcal{Q}_k$ of subcubes within Q by

$$\mathcal{Q}_k = \{T_\alpha Q'\colon \alpha \in \Gamma, Q' \in \mathcal{Q}_{k-1}\}.$$

By a *chain* of cubes we will mean a sequence (Q^i) such that $Q^i \in \mathcal{Q}_i$ and $Q^{i+1} \subset Q^i$ for each $i \in \mathbb{N}$. If $x, x' \in Q^i$, our definitions imply that $|x_\alpha^*(x - x')| \leq 2\varepsilon^i$ for all $\alpha \in \Gamma$, and thus $\bigcap_i Q^i$ consists of at most one element for each chain (Q^i). To see that every such intersection is indeed nonempty, note that the relation $Q^i = T_{\alpha_0} T_{\alpha_1} \cdots T_{\alpha_{i-1}} Q$ defines a bijective correspondence between chains and integer sequences $(\alpha_i) \in \Gamma^{\mathbb{N}}$. In terms of this correspondence, we have $5 \cdot \sum_{i=1}^k \varepsilon^i x_{\alpha_{i-1}} \in Q^k$ for each k; by our choice of ε, this series converges in norm to a vector in $B_X(1)$.

Let $h_\alpha = h \circ T_\alpha^{-1}$, where the function $h\colon X \to \mathbb{R}$ is defined as in Section 1. Each of the h_α is then smooth, and their supports are disjoint and lie within Q. Choose a dense subset $(y_\alpha)_{\alpha\in\Gamma}$ in the unit ball of Y and define

$$f(x) = \sum_{\alpha\in\Gamma} h_\alpha(x) \cdot y_\alpha.$$

Clearly f is C^∞, $f(Q_\alpha) = y_\alpha$. Moreover, Lemma 1.2 implies that the $\mathbf{C}^k$ norm of $f|_{B_X(n)}$, denoted $|f|_{k,n}$, is finite for every choice of $k, n \in \mathbb{N}$, and we may choose a sequence of positive constants (δ_k) such that $\sum_k \delta_k \varepsilon^{-k^2} |f|_{k,4\varepsilon^{-k}}$ converges.

Next, let $\mathcal{U}_k$ denote the union of all members of $\mathcal{Q}_k$, and define a sequence of mappings g_k: $X \to Y$ by setting $g_0 = f$,

$$g_k(x) = \begin{cases} g_{k-1}(x) & \text{if } x \notin \mathcal{U}_k \\ (S \circ f \circ T^{-1})(x) & \text{if } x \in Q' \in \mathcal{Q}_k \end{cases}$$

where S is the affine dilation mapping $B_Y(1)$ onto the ball $B_Y(p, \delta_k)$ of radius δ_k centered at $p = g_{k-1}(Q')$, and T now denotes the affine map sending the cube Q onto Q'. (Note that since f is by definition constant on each $Q \in \mathcal{Q}_1$, it follows that for every k, the map g_{k-1} is constant on each $Q \in \mathcal{K}_k$.)

We claim that the mappings g_k converge to a C^∞ mapping $X \to Y$. By construction, every point of $X \setminus \bigcap_k \mathcal{U}_k$ has a neighborhood on which all but finitely many of the g_k coincide; since $\bigcap_k \mathcal{U}_k \subset B_X(1)$, our claim reduces to proving convergence on $B_X(2)$. To this end, note that since $B_X(2) \subset T(B_X(4\varepsilon^{-k}))$, we have the estimate

$$|g_k - g_{k-1}|_{k,2} \le \delta_k \varepsilon^{-k^2} |f|_{k,4\varepsilon^{-k}}$$

and by our choice of scaling constants δ_k, the sequence (g_k) must C^∞-converge on $B_X(2)$ to a smooth map g: $B_X(2) \to Y$. Consequently, the map $g = \lim g_k$ is defined and smooth on all of X. Additionally, boundedness of $|g|_{1,R}$ for all $R > 0$ implies that g is Lipschitz on bounded subsets of X. (Since the basic cube Q is unbounded, g does not have bounded support and is not Lipschitz on all of X).

To see that g maps onto $B_Y(1)$, first note that for each $Q' \in \mathcal{Q}_{k-1}$, the set

$$\{g(\partial Q''): Q'' \in \mathcal{Q}^k \text{ and } Q'' \subset Q'\}$$

is dense in $B_Y(p, \delta_k)$, where $p = g(\partial Q')$. Given $y \in B_Y(1)$, we apply this observation inductively to choose a chain (Q^i) such that for each i, $|y - p_i| \le \delta_{i+1}$, where $p_i = g(\partial Q^i)$. If x is the unique element of the intersection $\bigcap_i Q^i$, then by continuity $g(x) = y$.

In order to map X onto all of Y, first observe that $g = 0$ outside of Q. For any fixed $\alpha \in \Gamma$, the mappings $f_n(x) = n \cdot g(x - 4nx_\alpha)$ have pairwise disjoint supports and map onto $B_Y(n)$, respectively. Thus, the map F: $X \to Y$ given by $F = \sum_n f_n$ is a C^∞ surjection. ∎

Proof of Theorem 3(1) (Sketch): Suppose that X is a Banach space satisfying condition $(\mathcal{K})_1$, so that there exist a bounded collection $(z_\alpha)_{\alpha\in\Gamma} \subset X$, a Banach space X' with C^1 Lipschitz bump functions, and a bounded linear map $T: X \to X'$ such that the vectors $T(z_\alpha)$ are 1-separated in X'. Choose $r > 1$ such that $\|Tz_\alpha\| < r/2$ for all $\alpha \in \Gamma$. Using any C^1 Lipschitz bump functions on X' it is easy to construct a C^1 Lipschitz function $s: X' \to \mathbb{R}$ such that $s = 1$ on $B_X(r)$ and $s(x) = 0$ whenever $|x| > R$ for some $R > r$. Fix a positive constant $\varepsilon < (2R)^{-1}$ and for each $\alpha \in \Gamma$ let $L_\alpha: X' \to X'$ be the affine map $L_\alpha(x) = \varepsilon x + Tz_\alpha$.

Given any Banach space Y with $\mathrm{dens}(Y) \leq \mathcal{K}$, we can proceed exactly as in the proof of Theorem 3(2), substituting the Tz_α for the x_α, the function s for h, the L_α for the T_α, etc., to obtain a C^1 map $g: X' \to Y$ which transforms the set of vectors of the form $\sum_{i=1}^\infty \varepsilon^i T z_{\alpha_{i-1}} = T\left(\sum_{i=1}^\infty \varepsilon^i z_{\alpha_{i-1}}\right)$ for $\alpha_{i-1} \in \Gamma$ onto a ball in Y. (The Lipschitz property of s is needed to insure the C^1 convergence of the g_k constructed as before.) Moreover, g has bounded support and is therefore Lipschitz on X', so that the composition $(g \circ T): X \to Y$ is a C^1 Lipschitz map onto a neighborhood of the origin in Y.

If $w \in X$ is any vector satisfying $\|Tw\| = 1$, then the mappings

$$f_n(x) = n \cdot (g \circ T)\left(\frac{x - 4^n R w}{n}\right)$$

are C^1, have the same Lipschitz constant as $g \circ T$, and have pairwise disjoint supports. Since $Y = \bigcup_n f_n(X)$, the function $F = \sum_n f_n$ is a C^1 Lipschitz surjection $X \to Y$. ∎

3. Proofs of Theorems 1 and 2

By the Joseffson-Nissenzweig theorem, every infinite-dimensional Banach space admits a non-compact bounded linear map into $c_0(\mathbb{N})$. Since $c_0(\mathbb{N})$ admits a C^1 bounded bump function, it follows that any infinite-dimensional Banach space satisfies condition $(\aleph_0)_1$, and so Theorem 3(1) implies Theorem 1.

Next, we show that the case $\mathcal{K} = \aleph_0$ of Theorem 3(2) is similarly equivalent to Theorem 2. The following argument was given to me by E. Odell and H. Rosenthal during a visit to U. T. Austin in April, 1993. We begin with a useful corollary of Ramsey's theorem (see [5]).

LEMMA 3.1: *For any semi-normalized Banach-Saks sequence (x_j) in a Banach space X there exists $(\lambda_n) \in c_0(\mathbb{N})$ and $(y_j) \subset (x_j)$ such that for any $n \le m_1 < m_2 < \cdots < m_n$,*

$$\Big\| \sum_{j=1}^{n} y_{m_j} \Big\| \le n\lambda_n.$$

LEMMA 3.2: *A Banach space X satisfies condition $(\aleph_0)_\infty$ if and only if X^* contains a normalized, weakly null Banach-Saks sequence.*

Proof: Suppose X^* contains a normalized, weakly null Banach-Saks sequence (x_j^*), and choose $(\lambda_j) \in c_0(\mathbb{N})$ and a subsequence $(y_j^*) \subset (x_j^*)$ as in Lemma 3.1. Given $\varepsilon > 0$, choose n such that $\lambda_n < \varepsilon$ and set $k(\varepsilon) = 3n$. Now suppose $x \in B_X(1)$ contradicts this choice, so that for some $\varepsilon > 0$, we have

$$\operatorname{card}\{j\colon |y_j^*(x)| \ge \varepsilon\} > k(\varepsilon).$$

Then this set contains n distinct indices $m_j > n$ such that the corresponding numbers $y_{m_j}^*(x)$ have the same sign. This implies

$$\Big\| \sum_{j=1}^{n} y_{m_j}^* \Big\| \ge n\varepsilon > n\lambda_n.$$

On the other hand, our choice of (y_j^*) and (λ_j) requires that

$$\Big\| \sum_{j=1}^{n} y_{m_j}^* \Big\| < n\lambda_n,$$

and we arrive at a contradiction. Thus, X satisfies condition $(\aleph_0)_\infty$.

To prove the converse, let $(x_j^*)_{j\in\mathbb{Z}_+} \subset X^*$ be a normalized sequence with the property that for any $\varepsilon > 0$, there exists $k(\varepsilon)$ such that for any $x \in B_X(1)$, we have $\operatorname{card}\{j \in \mathbb{Z}_+\colon |x_j^*(x)| > \varepsilon\} \le k(\varepsilon)$. If (x_j^*) is not weakly null, then there exists $\varepsilon > 0$ and $\lambda \in B_{X^{**}}(1)$ such that $|\lambda(x_j^*)| > 2\varepsilon$ for infinitely many indices j. By Goldstine's theorem, there then exists vector $x \in B_X(1)$ such that $|x_j^*(x)| \ge \varepsilon$ for more than $k(\varepsilon)$ indices $j \in \mathbb{Z}_+$, which contradicts the definition of $k(\varepsilon)$.

Next, fix an arbitrary subsequence (y_j^*) of (x_j^*) and define

$$\beta_n = \Big\| \sum_{j=1}^{n} y_j^* \Big\|.$$

For any $x \in B_X(1)$ we therefore have

$$\Big|\sum_{j=1}^{n} y_j^*(x)\Big| \leq n\varepsilon + k(\varepsilon),$$

and so

$$\beta_n \leq n\varepsilon + k(\varepsilon).$$

Consequently,

$$\limsup_{n\to\infty} \frac{\beta_n}{n} \leq \varepsilon.$$

Since $\varepsilon > 0$ and (y_j^*) were arbitrary, the sequence (x_j^*) is Banach-Saks. ∎

4. Concluding remarks

In this section, we focus on some separable spaces illustrating the scope of our construction. Reflexive spaces whose duals possess the Banach-Saks property (see [5]) provide a large class of examples satisfying the hypotheses of Theorem 2. These include all reflexive B-convex spaces and the dual of any reflexive stable space. It is interesting to note that in fact condition $(\mathcal{K})_\infty$ can be checked directly in any super-reflexive space of density character $\mathcal{K}$ using the theorem of Guarii–Guarii [8] and James [9] and the existence of (transfinite) basic sequences. Finally, Theorem 2 applies to the James space and its dual, as well as the Tsirelson space, the Schreier space (see [6]), and $\ell_1(\mathbb{N})$.

Turning to pathologies, we note that a dichotomy due to Rosenthal [12] (see [5, Prop. 2, p. 58]) implies

COROLLARY 4.1: *Let X be a separable, infinite-dimensional Banach space. Then X fails the hypotheses of Theorem 2 if and only if at least one of the following holds:*

1. *Every semi-normalized weakly null sequence in X^* has a subsequence with a spreading model isomorphic to $\ell_1(\mathbb{N})$.*
2. *X^* has the Schur property.*

Case (1) of this corollary is illustrated by $X^* = T$, the Tsirelson space, and thus our construction does not apply to every reflexive Banach space. Similarly, the sequence space $c_0(\mathbb{N})$ fails Theorem 2 since its dual $\ell_1(\mathbb{N})$ has the Schur property. It would be interesting to determine whether there exist separable spaces X, Y

such that the image of every smooth map $X \to Y$ has empty interior. For example, an answer to the following question appears to be unknown

QUESTION 4.2: *Does there exist a C^∞ map of $c_0(\mathbb{N})$ onto an open subset of the Hilbert space $\ell_2(\mathbb{N})$?*

References

[1] S. M. Bates, *A smooth surjective rank-1 endomorphism of a Hilbert space*, International Mathematics Research Notices **6** (1991), 71–73.

[2] S. M. Bates, *On the image size of singular maps II.*, Duke Mathematical Journal **68** (1992), 463–476.

[3] S. M. Bates, *Lipschitz mappings of Banach-Carnot groups*, preprint, 1995.

[4] S. M. Bates and C. C. Pugh, *Super singular surjections*, Pacific Journal of Mathematics, to appear.

[5] B. Beauzamy and J.-T. Lapresté, *Modèles étalés des espaces de Banach*, Travaux en Cours, Hermann, Paris, 1984.

[6] P. Casazza and T. Shura, *Tsirelson's space*, Lecture Notes in Mathematics **1363**, Springer-Verlag, Berlin, 1989.

[7] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific, 1993.

[8] V. I. Gurarii and N. I. Gurarii, *On bases in uniformly convex and uniformly smooth Banach spaces*, Izvestiya Akademii Nauk SSSR **35** (1971), 210–215.

[9] R. C. James, *Superreflexive spaces with bases*, Pacific Journal of Mathematics **41** (1972), 409–419.

[10] M. I. Kadec, *On topological equivalence of separable Banach spaces*, Doklady Adakemii Nauk SSSR **162** (1965), 23–25; English translation: Soviet Mathematics Doklady **7** (1966) 319–322.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, Berlin, 1977.

[12] H. Rosenthal, *Weakly independent sequences and the Banach-Saks property*, The Bulletin of the London Mathematical Society **8** (1976), 22–24.

[13] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[14] H. Torunczyk, *Characterizing Hilbert space topology*, Fundamenta Mathematicae **111** (1981), 247–262.

[15] Y. Yomdin, *Surjective mappings whose differential is nowhere surjective*, Proceedings of the American Mathematical Society **111** (1991), 267–270.